

BLOW-UP BEHAVIOR OF THE SCALAR CURVATURE ALONG THE CONICAL KÄHLER-RICCI FLOW WITH FINITE TIME SINGULARITIES

RYOSUKE NOMURA

ABSTRACT. We investigate the scalar curvature behavior along the normalized conical Kähler-Ricci flow ω_t , which is the conic version of the normalized Kähler-Ricci flow, with finite maximal existence time $T < \infty$. We prove that the scalar curvature of ω_t is bounded from above by $C/(T-t)^2$ under the existence of a contraction associated to the limiting cohomology class $[\omega_T]$. This generalizes Z. Zhang's work to the conic case.

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1. INTRODUCTION

Let X be a compact Kähler manifold of dimension n , D be a smooth divisor on X , and β be a positive real number satisfying $0 < \beta < 1$. We consider the normalized conical Kähler-Ricci flow ω_t on X which is a family of cone metrics with cone angle $2\pi\beta$ along D satisfying the following parabolic equation:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} \omega_t &= -\text{Ric}(\omega_t) - \omega_t + 2\pi(1-\beta)[D], \\ \omega_t|_{t=0} &= \omega^*, \end{cases}$$

where $[D]$ is the current of integration over D , and ω^* is a certain initial cone metric defined later (see (1.2)). In the case of $D = 0$, ω_t is called the normalized Kähler-Ricci flow. This case has been studied extensively in the past decades (see [TZ06, ST09, ST12, CW12, BEG13, CW14a, CSW15, CT15, GSW15] and the references therein).

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The maximal existence time T of the normalized conical Kähler-Ricci flow ω_t is characterized by the following cohomological condition:

$$T = \sup\{t > 0 \mid [\omega_t] = e^{-t}[\omega_0] + (1 - e^{-t})2\pi c_1(K_X + (1 - \beta)D) \text{ is a Kähler class}\}.$$

In particular, the limiting class $[\omega_T]$ is nef but not Kähler. This characterization is due to Shen [She14b, She14a]. As t tends to T , the flow ω_t might form singularities. The analysis of the singularities, in particular its curvature behavior, is one of the main objects in the study of the geometric flows. Our purpose here is to investigate the scalar curvature behavior of ω_t with finite time singularities (i.e. $T < \infty$) as t approaches to T .

In the infinite time singularities case (i.e. $T = \infty$), the uniform boundedness of the scalar curvature of the normalized Kähler-Ricci flow (i.e. $D = 0$) was proved by Zhang [Zha09] when K_X is nef and big. This result was extended by Song-Tian [ST11] when K_X is semi-ample. Furthermore, Edwards [Edw15] generalized these results to the conic setting. In the case of Fano manifolds (i.e. $-K_X$ is ample), Perelman established a uniform boundedness of it (see [ST08]) and Liu-Zhang [LZ14] extended to the conic case.

On the other hand, in the finite time singularities case, Collins and Tosatti [CT15] proved that the scalar curvature of the normalized Kähler-Ricci flow ω_t (i.e. $D = 0$) blows up precisely along the null locus of $[\omega_T]$. Zhang [Zha10] showed that the scalar curvature $R(\omega_t)$ of the normalized Kähler-Ricci flow ω_t satisfies

$$R(\omega_t) \leq \frac{C}{(T - t)^2}$$

assuming the semi-ampleness of $[\omega_T]$. This condition is natural in terms of the deep relationship between the Kähler-Ricci flow and the minimal model program (see [ST09, Zha10]). Our main theorem generalizes this to the conic setting.

We assume the following contraction type condition on the cohomology class $[\omega_T]$. Let $f: X \rightarrow Z$ be a holomorphic map between compact Kähler manifolds, whose image is contained in a normal irreducible subvariety Y of Z . Let D_Y be an effective Cartier divisor on Y such that the pullback of D_Y satisfies $D = f^*D_Y$. Let h_Y be a smooth Hermitian metric on the line bundle $\mathcal{O}_Y(D_Y)$ in the sense of [EGZ09, Section 5], and s_Y be a holomorphic section of $\mathcal{O}_Y(D_Y)$ whose zero divisor is D_Y . We define the initial cone metric ω^* by

$$(1.2) \quad \omega^* := \omega_0 + k\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta},$$

where ω_0 is a smooth Kähler form on X , $k \in \mathbb{R}_{>0}$ is a sufficiently small real number, $s := f^*s_Y$ is the holomorphic section of $\mathcal{O}_X(D)$, and $h := f^*h_Y$ is the smooth Hermitian metric on $\mathcal{O}_X(D)$. We remark that if we take k sufficiently small, ω^* is actually a cone metric with cone angle $2\pi\beta$ along D .

Let ω_t be the normalized conical Kähler-Ricci flow with initial cone metric ω^* , and T be the maximal existence time of ω_t . We further assume that T is finite and there exists a smooth Kähler form ω_Z on Z satisfying

$$[f^*\omega_Z] = [\omega_T] \in H^{1,1}(X, \mathbb{R}).$$

Under these assumptions, we have the following theorem.

Theorem A. *The scalar curvature $R(\omega_t)$ of ω_t satisfies*

$$R(\omega_t) \leq \frac{C}{(T-t)^2} \quad \text{on } X \setminus D,$$

where $C > 0$ is a constant independent of t .

In contrast with Zhang's result, we need to treat with the singularities of ω_t along D . This is overcome by using the approximation technique used in [CGP13, She14b, LZ14, Edw15].

Remark 1.1. If we replace $(1-\beta)D$ by $\sum_{i \in I} (1-\beta_i)D_i$ where D_i are smooth divisors intersecting transversely, the same argument below gives the same conclusion. But for simplicity, we only treat one smooth divisor case.

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2. APPROXIMATION OF THE NORMALIZED CONICAL KÄHLER-RICCI FLOW BY THE TWISTED NORMALIZED KÄHLER-RICCI FLOW

In the following argument, we assume that the conditions in Theorem A are always satisfied. We first define a family of reference smooth Kähler forms $\widehat{\omega}_t$ whose cohomology classes are equal to $[\omega_t]$. We set $\widehat{\omega}_\infty$ by

$$\begin{aligned} \widehat{\omega}_\infty &:= -\frac{e^{-T}}{1-e^{-T}}\omega_0 + \frac{1}{1-e^{-T}}f^*\omega_Z \\ &\in -\frac{e^{-T}}{1-e^{-T}}[\omega_0] + \frac{1}{1-e^{-T}}[\omega_T] = 2\pi c_1(K_X + (1-\beta)D), \end{aligned}$$

and $\widehat{\omega}_t$ by

$$(2.1) \quad \widehat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\widehat{\omega}_\infty = a_t\omega_0 + (1-a_t)\widehat{\omega}_T,$$

where $a_t := (e^{-t} - e^{-T})/(1 - e^{-T})$. In this setting, $\widehat{\omega}_T = f^*\omega_Z \geq 0$ is semi-positive, hence $\widehat{\omega}_t$ are smooth Kähler forms for any $t \in [0, T)$. The cohomology class of $\widehat{\omega}_t$ coincide with $[\omega_t]$.

We next define a family of reference smooth Kähler forms $\widetilde{\omega}_{\varepsilon,t}$ whose cohomology classes are equal to $[\omega_t]$. We use the approximation method as in [She14b, LZ14, Edw15] originated from [CGP13]. We denote $\rho_\varepsilon := \chi(|s|_h^2, \varepsilon^2)$, where

$$\chi(u, \varepsilon^2) := \beta \int_0^u \frac{(r + \varepsilon^2)^\beta - \varepsilon^{2\beta}}{r} dr.$$

Then, ρ_ε are smooth functions on X and converge to $|s|_h^{2\beta}$ in $C_{\text{loc}}^\infty(X \setminus D)$ as $\varepsilon \rightarrow 0$. In this notation, we define reference smooth Kähler forms $\widetilde{\omega}_{\varepsilon,t}$ by

$$(2.2) \quad \widetilde{\omega}_{\varepsilon,t} := \widehat{\omega}_t + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon = a_t\widetilde{\omega}_{\varepsilon,0} + (1-a_t)\widetilde{\omega}_{\varepsilon,T}.$$

These forms converge to $\widetilde{\omega}_t$ in $C_{\text{loc}}^\infty(X \setminus D)$ and as current on X when ε tends to 0.

We prove that if we take k sufficiently small, $\widetilde{\omega}_{\varepsilon,t}$ is positive for all $t \in [0, T)$. Let $C_1 > 0$ be a constant satisfying

$$(2.3) \quad -C_1\omega_Z \leq \sqrt{-1}R_{h_Y} \leq C_1\omega_Z \quad \text{on } Y,$$

where R_{h_Y} is the Chern curvature of h_Y . Since $h = f^*h_Y$ and $\widehat{\omega}_T = f^*\omega_Z$, we have

$$(2.4) \quad -C_1\widehat{\omega}_T \leq \sqrt{-1}R_h \leq C_1\widehat{\omega}_T \quad \text{on } X.$$

Let $C_2 > 0$ and $C_3 > 1$ be constants such that

$$(2.5) \quad \sup_Y |s_Y|_{h_Y} \leq C_2,$$

$$(2.6) \quad \widehat{\omega}_T = f^*\omega_Z \leq C_3\omega_0 \quad \text{on } X.$$

By (2.5), there exists a constant $C_4 > 0$ independent of ε such that

$$(2.7) \quad 0 \leq \rho_\varepsilon \leq C_4 \quad \text{on } X.$$

By the computation in [CGP13, Section 3], we have

$$(2.8) \quad \begin{aligned} \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon &= \beta^2 \frac{\sqrt{-1}\langle \nabla s \wedge \nabla \bar{s} \rangle_h}{(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \beta(|s|_h^2 + \varepsilon^2)^\beta - \varepsilon^{2\beta} \sqrt{-1}R_h \\ &\geq -\beta C_1 C_2^{2\beta} \widehat{\omega}_T, \end{aligned}$$

where ∇ is the Chern connection of the line bundle $(\mathcal{O}_X(D), h)$, R_h is its Chern curvature, and $\sqrt{-1}\langle \nabla s \wedge \nabla \bar{s} \rangle_h$ is a semi-positive closed real $(1, 1)$ -form combining the wedge product of differential forms with the Hermitian metric h on $\mathcal{O}_X(D)$. By (2.2), (2.8), and (2.6), we obtain the following inequalities:

$$(2.9) \quad \widetilde{\omega}_{\varepsilon, T} = \widehat{\omega}_T + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \geq (1 - k\beta C_1 C_2^{2\beta})\widehat{\omega}_T \geq (1 - k\beta C_1 C_2^{2\beta} C_3)\widehat{\omega}_T,$$

$$(2.10) \quad \widetilde{\omega}_{\varepsilon, 0} = \omega_0 + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \geq \omega_0 - k\beta C_1 C_2^{2\beta} \widehat{\omega}_T \geq (1 - k\beta C_1 C_2^{2\beta} C_3)\omega_0.$$

Finally, these inequalities give the positivity of $\widetilde{\omega}_{\varepsilon, t}$ for any $t \in [0, T]$:

$$\widetilde{\omega}_{\varepsilon, t} = \widehat{\omega}_t + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon = a_t \widetilde{\omega}_{\varepsilon, 0} + (1 - a_t) \widetilde{\omega}_{\varepsilon, T} \geq (1 - k\beta C_1 C_2^{2\beta} C_3) \widehat{\omega}_t > 0.$$

By using these reference smooth Kähler forms, we consider the following approximate flow:

$$(2.11) \quad \begin{cases} \frac{\partial}{\partial t} \varphi_{\varepsilon, t} &= \log \frac{(\widetilde{\omega}_{\varepsilon, t} + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon, t})^n}{\Omega} - \varphi_{\varepsilon, t} + (1 - \beta) \log(|s|_h^2 + \varepsilon^2) - k\rho_\varepsilon, \\ \varphi_{\varepsilon, t}|_{t=0} &= 0, \end{cases}$$

where Ω is a smooth volume form on X satisfying

$$-\text{Ric}(\Omega) + (1 - \beta)\sqrt{-1}R_h = \widehat{\omega}_\infty \in 2\pi c_1(K_X + (1 - \beta)D).$$

We set $\omega_{\varepsilon, t}$ by

$$(2.12) \quad \omega_{\varepsilon, t} := \widetilde{\omega}_{\varepsilon, t} + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon, t}.$$

Then, $\omega_{\varepsilon, t}$ satisfies the following twisted Kähler-Ricci flow:

$$(2.13) \quad \begin{cases} \frac{\partial}{\partial t} \omega_{\varepsilon, t} &= -\text{Ric}(\omega_{\varepsilon, t}) - \omega_{\varepsilon, t} + \eta_\varepsilon, \\ \omega_t|_{t=0} &= \widetilde{\omega}_{\varepsilon, 0} (:= \omega_0 + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon), \end{cases}$$

where η_ε is a closed real $(1, 1)$ -form defined by $\eta_\varepsilon := (1 - \beta)\sqrt{-1}\partial\bar{\partial} \log(|s|_h^2 + \varepsilon^2) + (1 - \beta)\sqrt{-1}R_h$. η_ε converges to $2\pi(1 - \beta)[D]$ in $C_{\text{loc}}^\infty(X \setminus D)$ and as current on X when ε goes to 0.

The validity of these approximations (2.11), (2.13) is justified by the following theorem due to Shen [She14b].

Theorem 2.1 ([She14b, Section 2]). *There exists a subsequence ε_i converging to 0 as $i \rightarrow \infty$ such that $\varphi_{\varepsilon_i, t}$ converges to φ_t in $C_{\text{loc}}^\infty(X \setminus D)$ and $\omega_{\varepsilon_i, t}$ converges to ω_t in $C_{\text{loc}}^\infty(X \setminus D)$ and as current on X .*

Thanks to this theorem, we only need to estimate $\varphi_{\varepsilon, t}$ and $\omega_{\varepsilon, t}$.

3. OVERVIEW OF THE PROOF OF THEOREM A

In this section, we outline the proof of Theorem A. First, we need the following formulas.

Proposition 3.1. *The Ricci curvature $\text{Ric}(\omega_{\varepsilon, t})$ and the scalar curvature $R(\omega_{\varepsilon, t})$ satisfy the following formulas:*

$$\begin{aligned} (a) \quad & (1 - e^{t-T}) (\text{Ric}(\omega_{\varepsilon, t}) - \eta_\varepsilon) = -\sqrt{-1} \partial \bar{\partial} v_{\varepsilon, t} + e^{t-T} \omega_{\varepsilon, t} - \widehat{\omega}_T, \\ (b) \quad & (1 - e^{t-T}) (R(\omega_{\varepsilon, t}) - \text{tr}_{\omega_{\varepsilon, t}}(\eta_\varepsilon)) = -\Delta_{\omega_{\varepsilon, t}} v_{\varepsilon, t} + n e^{t-T} - \text{tr}_{\omega_{\varepsilon, t}}(\widehat{\omega}_T), \end{aligned}$$

where

$$v_{\varepsilon, t} := (1 - e^{t-T}) \dot{\varphi}_{\varepsilon, t} + \varphi_{\varepsilon, t} + k \rho_\varepsilon.$$

Proof. (b) follows from (a) by taking traces. We prove (a). By (2.13), (2.2), and (2.12), we have

$$\begin{aligned} \text{Ric}(\omega_{\varepsilon, t}) - \eta_\varepsilon &= -\frac{\partial}{\partial t} \omega_{\varepsilon, t} - \omega_{\varepsilon, t} \\ &= -\left(\frac{\partial}{\partial t} \widehat{\omega}_t + \frac{\partial}{\partial t} \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon, t} \right) - (\widehat{\omega}_t + k \sqrt{-1} \partial \bar{\partial} \rho_\varepsilon + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon, t}) \\ &= -\sqrt{-1} \partial \bar{\partial} (\dot{\varphi}_{\varepsilon, t} + \varphi_{\varepsilon, t} + k \rho_\varepsilon) - \left(\widehat{\omega}_t + \frac{\partial}{\partial t} \widehat{\omega}_t \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} -e^{t-T} (\text{Ric}(\omega_{\varepsilon, t}) - \eta_\varepsilon) &= -e^{t-T} \left(-\frac{\partial}{\partial t} \omega_{\varepsilon, t} - \omega_{\varepsilon, t} \right) \\ &= e^{t-T} \left(\frac{\partial}{\partial t} \widehat{\omega}_t + \frac{\partial}{\partial t} \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon, t} \right) + e^{t-T} \omega_{\varepsilon, t} \\ &= \sqrt{-1} \partial \bar{\partial} (e^{t-T} \dot{\varphi}_{\varepsilon, t}) + e^{t-T} \omega_{\varepsilon, t} + e^{t-T} \frac{\partial}{\partial t} \widehat{\omega}_t. \end{aligned}$$

Combining these equalities and (2.1), we have (a). \square

By this proposition, to obtain the upper bound for the scalar curvature $R(\omega_{\varepsilon, t})$, we only need to estimate $u_{\varepsilon, t} := \text{tr}_{\omega_{\varepsilon, t}}(\widehat{\omega}_T)$ and $\Delta_{\omega_{\varepsilon, t}} v_{\varepsilon, t}$. We devide our argument into the following 5 steps:

Step 1. The C^0 -estimate for $v_{\varepsilon, t}$ (Section 4).

Step 2. The C^0 -estimate for $u_{\varepsilon, t} := \text{tr}_{\omega_{\varepsilon, t}}(\widehat{\omega}_T)$ using Step 1 and the parabolic Schwarz lemma (Section 5).

Step 3. The gradient estimate for $v_{\varepsilon, t}$ (Section 6).

Step 4. The Laplacian estimate for $v_{\varepsilon, t}$ (Section 7):

$$\Delta_{\omega_{\varepsilon, t}} v_{\varepsilon, t} \geq -\frac{C}{T-t}.$$

Step 5. Proof of Theorem A (Section 7).

4. THE C^0 -ESTIMATE FOR $v_{\varepsilon,t}$

In this section, we prove the C^0 -estimates for $v_{\varepsilon,t}$. More precisely, we prove the following proposition.

Proposition 4.1. *There exists a constant $C_5 > 0$ independent of ε and t such that*

$$\|v_{\varepsilon,t}\|_{C^0} \leq C_5$$

holds.

To apply the maximum principle, we need the following lemma.

Lemma 4.2. *$v_{\varepsilon,t}$ satisfies the following evolution equation*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) v_{\varepsilon,t} = -n + u_{\varepsilon,t},$$

where $u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)$.

Proof. Differentiating (2.11) with respect to t , we have

$$(4.1) \quad \begin{aligned} \frac{\partial}{\partial t} \dot{\varphi}_{\varepsilon,t} &= \text{tr}_{\omega_{\varepsilon,t}} \left(\frac{\partial}{\partial t} (\widetilde{\omega}_{\varepsilon,t} + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon,t}) \right) - \dot{\varphi}_{\varepsilon,t} \\ \text{i.e. } \frac{\partial}{\partial t} (\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t}) &= \text{tr}_{\omega_{\varepsilon,t}} \left(\frac{\partial}{\partial t} \widehat{\omega}_t \right) + \Delta_{\omega_{\varepsilon,t}} \dot{\varphi}_{\varepsilon,t}. \end{aligned}$$

On the other hand, by (2.12) and (2.2), we have

$$\Delta_{\omega_{\varepsilon,t}} \varphi_{\varepsilon,t} = \text{tr}_{\omega_{\varepsilon,t}}(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\widetilde{\omega}_{\varepsilon,t}) = n - \text{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_t) - \Delta_{\omega_{\varepsilon,t}}(k\rho_{\varepsilon}).$$

Combing these, we obtain

$$(4.2) \quad \frac{\partial}{\partial t} (\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\sqrt{-1} \partial \bar{\partial} \rho_{\varepsilon}) = \Delta_{\omega_{\varepsilon,t}} (\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon}) - n + \text{tr}_{\omega_{\varepsilon,t}} \left(\widehat{\omega}_t + \frac{\partial}{\partial t} \widehat{\omega}_t \right).$$

Next, by using (4.1), we have

$$(4.3) \quad \frac{\partial}{\partial t} (-e^{t-T} \dot{\varphi}_{\varepsilon,t}) = -e^{t-T} \frac{\partial}{\partial t} (\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t}) = -\text{tr}_{\omega_{\varepsilon,t}} \left(e^{t-T} \frac{\partial}{\partial t} \widehat{\omega}_t \right) - \Delta_{\omega_{\varepsilon,t}} (e^{t-T} \dot{\varphi}_{\varepsilon,t}).$$

By (4.2), (4.3), and (2.1), we get the assertion. \square

Next, we prove the uniform volume estimate of the reference metrics $\widetilde{\omega}_{\varepsilon,t}$.

Lemma 4.3. *There exists a constant $C_6 > 0$ independent of ε and t satisfying the following inequalities:*

$$(a) \quad \frac{1}{C_6} \frac{\Omega}{(|s|_h^2 + \varepsilon^2)^{1-\beta}} \leq \widetilde{\omega}_{\varepsilon,0}^n \leq C_6 \frac{\Omega}{(|s|_h^2 + \varepsilon^2)^{1-\beta}}.$$

$$(b) \quad \widetilde{\omega}_{\varepsilon,t}^n \leq C_3^m C_6 \frac{\Omega}{(|s|_h^2 + \varepsilon^2)^{1-\beta}}.$$

Proof. The first inequality follows from (2.8). We prove the second one. For $0 < k < C_3$, by (2.4) and (2.2), we have

$$\tilde{\omega}_{\varepsilon,T} = \hat{\omega}_T + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \leq C_3\omega_0 + k\sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \leq C_3\tilde{\omega}_{\varepsilon,0}.$$

Since $C_3 > 1$, we have

$$\tilde{\omega}_{\varepsilon,t} = a_t\tilde{\omega}_{\varepsilon,0} + (1 - a_t)\tilde{\omega}_{\varepsilon,T} \leq a_t\tilde{\omega}_{\varepsilon,0} + C_3(1 - a_t)\tilde{\omega}_{\varepsilon,0} \leq C_3\tilde{\omega}_{\varepsilon,0}.$$

Therefore, we get the assertion. \square

Using these lemmas, we can prove the uniform lower boundedness of $v_{\varepsilon,t}$.

Proposition 4.4. *$v_{\varepsilon,t}$ is uniformly lower bounded. More precisely, there exists a constant $C_7 > 0$ independent of ε and t such that*

$$v_{\varepsilon,t} \geq -C_7.$$

Proof. By Lemma 4.2 and the semi-positivity of $\hat{\omega}_T$, we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)(v_{\varepsilon,t} + nt) = u_{\varepsilon,t} = \text{tr}_{\omega_{\varepsilon,t}}(\hat{\omega}_T) \geq 0.$$

Thus, the maximum principle for $v_{\varepsilon,t} + nt$ gives the following:

$$v_{\varepsilon,t} + nt \geq \min_{X \times \{0\}} (v_{\varepsilon,t} + nt) = (1 - e^{-T})\dot{\varphi}_{\varepsilon,0} + k\rho_\varepsilon \geq (1 - e^{-T})\dot{\varphi}_{\varepsilon,0}.$$

Lemma 4.3 (a) and (2.7) give the lower boundedness of right hand side as follows:

$$\dot{\varphi}_{\varepsilon,0} = \log \frac{\tilde{\omega}_{\varepsilon,0}^n}{\Omega/(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,0} - k\rho_\varepsilon \geq -\log C_6 - kC_4.$$

Therefore we get the assertion. \square

To prove the uniform upper boundedness of $v_{\varepsilon,t} = (1 - e^{t-T})\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon$, it is enough to show that $\varphi_{\varepsilon,t}$ and $\dot{\varphi}_{\varepsilon,t}$ are uniformly upper bounded.

Proposition 4.5. *We have the following inequalities:*

- (a) $\varphi_{\varepsilon,t} \leq C_8$,
- (b) $\dot{\varphi}_{\varepsilon,t} \leq C_9$,

where $C_8 > 0$, $C_9 > 0$ independent of ε and t .

Proof. (a) Since $\varphi_{\varepsilon,0} = 0$, we may assume that $\varphi_{\varepsilon,t}$ takes maximum at $(x_0, t_0) \in X \times (0, T)$. By Lemma 4.3 (b), we have the following inequatliy which holds at (x_0, t_0) :

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t}\varphi_{\varepsilon,t} = \log \frac{(\tilde{\omega}_{\varepsilon,t} + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon,t})^n}{\Omega/(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,t} - k\rho_\varepsilon \\ &\leq \log \frac{\tilde{\omega}_{\varepsilon,t}^n}{\Omega/(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,t} - k\rho_\varepsilon \\ &\leq \log(C_3^n C_6) - \varphi_{\varepsilon,t}. \end{aligned}$$

Therefore, we obtain

$$\varphi_{\varepsilon,t}(x_0, t_0) \leq \log(C_3^n C_6) =: C_8.$$

Since (x_0, t_0) is arbitrary, $\varphi_{\varepsilon,t} \leq C_8$ holds on $X \times [0, T]$.

(b) We set $H_{\varepsilon,t} := (1 - e^t)\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon} + nt$. The same computation in Lemma 4.2 gives

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) H_{\varepsilon,t} = \text{tr}_{\omega_{\varepsilon,t}}(\omega_0) > 0.$$

By the maximum principle for $H_{\varepsilon,t}$, we have

$$H_{\varepsilon,t} \geq \min_{X \times \{0\}} H_{\varepsilon,t} = k\rho_{\varepsilon} \geq 0.$$

Therefore, combining with (a) and (2.7), we get the upper bound for $\dot{\varphi}_{\varepsilon,t}$:

$$\dot{\varphi}_{\varepsilon,t} \leq \frac{\varphi_{\varepsilon,t} + k\rho_{\varepsilon} + nt}{e^t - 1} \leq \frac{C_8 + kC_4 + nT}{e^t - 1}.$$

Combining with the uniform local estimate for the parabolic equation, we get the assertion. \square

5. THE C^0 -ESTIMATE FOR $u_{\varepsilon,t}$

In this section, we prove the following proposition.

Proposition 5.1. *There exists a constant $C_{10} > 0$ independent of ε and t such that*

$$0 \leq u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T) \leq C_{10}.$$

To prove this proposition, we need to estimate on $\eta_{\varepsilon} := (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log(|s|_h^2 + \varepsilon^2) + (1 - \beta)\sqrt{-1}R_h$ and the parabolic Schwartz lemma. A direct computation gives the following.

Lemma 5.2. *We have the following inequalities of η_{ε} .*

(a) *Lower boundedness of η_{ε} :*

$$\eta_{\varepsilon} = (1 - \beta) \frac{\varepsilon^2}{|s|_h^2 + \varepsilon^2} \left(\frac{\sqrt{-1}\langle \nabla s \wedge \nabla s \rangle_h}{|s|_h^2 + \varepsilon^2} + \sqrt{-1}R_h \right) \geq -(1 - \beta)C_1\widehat{\omega}_T.$$

(b) *For any Kähler form ω , we have*

$$-\langle \eta_{\varepsilon}, \widehat{\omega}_T \rangle_{\omega} \leq (1 - \beta)C_1|\widehat{\omega}_T|_{\omega}^2 \leq (1 - \beta)C_1(\text{tr}_{\omega}(\widehat{\omega}_T))^2.$$

By the fact that $\widehat{\omega}_T$ is the pullback of ω_Z by f , we can use the parabolic Schwarz lemma which is obtained by Song-Tian [ST07]. This is the parabolic version of [Yau78a].

Lemma 5.3 (parabolic Schwarz lemma). *$u_{\varepsilon,t}$ and $\log u_{\varepsilon,t}$ satisfy the following inequalities.*

(a)

$$\begin{aligned} \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} &\geq -C_Z u_{\varepsilon,t}^2 + \langle \text{Ric}(\omega_{\varepsilon,t}), \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} \\ &\geq -C_{11} u_{\varepsilon,t}^2 + \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}}. \end{aligned}$$

(b)

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) u_{\varepsilon,t} &\leq u_{\varepsilon,t} + C_Z u_{\varepsilon,t}^2 - \langle \eta_{\varepsilon}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}} \\ &\leq u_{\varepsilon,t} + C_{11} u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}}. \end{aligned}$$

$$\begin{aligned}
(c) \quad & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \log u_{\varepsilon,t} \leq C_Z u_{\varepsilon,t} + 1 - \frac{\langle \eta_{\varepsilon}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}}}{u_{\varepsilon,t}} \\
& \leq C_{11} u_{\varepsilon,t} + 1.
\end{aligned}$$

Here, ∇ is $(1,0)$ -part of the Levi-Civita connection of $\omega_{\varepsilon,t}$, $C_Z > 0$ is an upper bound for the bisectional curvature of ω_Z , and $C_{11} := C_Z + (1 - \beta)C_1 > 0$.

Proof of Proposition 5.1 We set $G_{\varepsilon,t} := \log u_{\varepsilon,t} - C_{12}v_{\varepsilon,t}$ where $C_{12} := C_{11} + 1 > 0$ is a uniform constant. The uniform upper boundedness of $G_{\varepsilon,0}$ follows from (2.6), (2.10) and Proposition 4.1. If we suppose that $G_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$, we have $u_{\varepsilon,t}(x_0, t_0) > 0$ and

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) G_{\varepsilon,t} &= \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \log u_{\varepsilon,t} - C_{12} \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) v_{\varepsilon,t} \\
&\leq (C_{11} u_{\varepsilon,t} + 1) - C_{12}(u_{\varepsilon,t} - n) \\
&= -u_{\varepsilon,t} + (C_{12}n + 1) \quad \text{at } (x_0, t_0).
\end{aligned}$$

By using the uniform boundedness of $v_{\varepsilon,t}$ (Proposition 4.1), we obtain

$$G_{\varepsilon,t} \leq \log(C_{12}n + 1) - C_{12}v_{\varepsilon,t} \leq \log(C_{12}n + 1) + C_{12}C_5 \quad \text{at } (x_0, t_0).$$

Since (x_0, t_0) is arbitrary, we have $G_{\varepsilon,t} \leq C_{13}$ on $X \times [0, T]$. Hence, using the definition of $G_{\varepsilon,t}$ and Proposition 4.1, we obtain

$$\log u_{\varepsilon,t} \leq C_{12}v_{\varepsilon,t} + C_{13} \leq C_{12}C_5 + C_{13},$$

which prove the assertion. \square

6. THE GRADIENT ESTIMATE FOR $v_{\varepsilon,t}$

In this section, we prove the following gradient estimate.

Proposition 6.1. *There exists a uniform constant $C_{14} > 0$ which is independent of ε and t such that*

$$|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \leq C_{14}.$$

To prove this proposition, as in [Zha10], we set $\Psi_{\varepsilon,t} := \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{A - v_{\varepsilon,t}}$, where $A > C_5 + 1$ is a fixed constant (see Proposition 4.1). We will use the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. The direct computation gives the following formulas.

Lemma 6.2. *We have the following formulas.*

$$\begin{aligned}
(a) \quad & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 = |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \eta_{\varepsilon}(\nabla v_{\varepsilon,t}, \overline{\nabla} v_{\varepsilon,t}) + 2\text{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\
& \quad - |\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - |\nabla \overline{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \\
(b) \quad & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} = \Delta_{\omega_{\varepsilon,t}}(v_{\varepsilon,t} + u_{\varepsilon,t}) + \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1}\partial\bar{\partial}v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}
\end{aligned}$$

$$\begin{aligned}
(c) \quad & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Psi_{\varepsilon,t} \\
&= \frac{1}{A - v_{\varepsilon,t}} \left(|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - |\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \eta_{\varepsilon}(\nabla v_{\varepsilon,t}, \bar{\nabla} v_{\varepsilon,t}) \right. \\
&\quad \left. + 2\operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \right) \\
&\quad + \frac{1}{(A - v_{\varepsilon,t})^2} \left((u_{\varepsilon,t} - n) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - 2\operatorname{Re}\langle \nabla |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \right) \\
&\quad - \frac{2}{(A - v_{\varepsilon,t})^3} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4
\end{aligned}$$

Proof of Proposition 6.1 We will apply the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. First, we estimate $\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Psi_{\varepsilon,t}$. By Lemma 5.2 (a), we have

$$-\eta_{\varepsilon}(\nabla v_{\varepsilon,t}, \bar{\nabla} v_{\varepsilon,t}) \leq (1 - \beta)C_1 |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2.$$

For sufficiently small constant $\delta > 0$ which will be determined later, we have

$$2\operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \leq 2|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}} |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}} \leq \frac{1}{\delta} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2.$$

Since

$$(6.1) \quad \nabla \Psi_{\varepsilon,t} = \frac{\nabla |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{A - v_{\varepsilon,t}} + \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{(A - v_{\varepsilon,t})^2} \nabla v_{\varepsilon,t},$$

we have

$$-\frac{2 - \delta}{(A - v_{\varepsilon,t})^2} \operatorname{Re}\langle \nabla |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} = -\frac{2 - \delta}{A - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + (2 - \delta) \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A - v_{\varepsilon,t})^3}.$$

On the other hand, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
|\langle \nabla |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}| &= \left| g^{i\bar{j}} g^{k\bar{l}} \left((\partial_k \partial_i v_{\varepsilon,t}) (\partial_{\bar{j}} v_{\varepsilon,t}) (\partial_{\bar{l}} v_{\varepsilon,t}) + (\partial_i v_{\varepsilon,t}) (\partial_k \partial_{\bar{j}} v_{\varepsilon,t}) (\partial_{\bar{l}} v_{\varepsilon,t}) \right) \right| \\
&\leq |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 (|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2) \\
&\leq \sqrt{2} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 (|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2)^{1/2}.
\end{aligned}$$

Therefore, we obtain the following:

$$\begin{aligned}
\frac{-\delta}{(A - v_{\varepsilon,t})^2} \operatorname{Re}\langle \nabla |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} &\leq \frac{\delta}{(A - v_{\varepsilon,t})^2} \left(\sqrt{2} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 (|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2)^{1/2} \right) \\
&= \sqrt{2} \delta \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{(A - v_{\varepsilon,t})^{3/2}} \frac{(|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2)^{1/2}}{(A - v_{\varepsilon,t})^{1/2}} \\
&\leq \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A - v_{\varepsilon,t})^3} + \delta \frac{|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{A - v_{\varepsilon,t}}.
\end{aligned}$$

Combining these inequalities with Proposition 5.1, Lemma 6.2 (c), and $A - v_{\varepsilon,t} > 1$, we obtain the following inequality:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Psi_{\varepsilon,t} \\ & \leq C_{15} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2-\delta}{A-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A-v_{\varepsilon,t})^3} \end{aligned}$$

where $C_{15} := 1 + (1-\beta)C_1 + (1/\delta) + C_{10} > 0$.

On the other hand, by Lemma 5.3 (a), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) u_{\varepsilon,t} \leq u_{\varepsilon,t} + C_{11} u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}} \leq C_{10} + C_{11} C_{10}^2 - 2\delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 =: C_{16} - 2\delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2$$

Here, we take $0 < \delta < 1/(2C_{10})$. Finally, we obtain the following:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) (\Psi_{\varepsilon,t} + u_{\varepsilon,t}) \\ & \leq \left(C_{15} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2-\delta}{A-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A-v_{\varepsilon,t})^3} \right) \\ & \quad + \left(C_{16} - 2\delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \right) \\ & = C_{16} + C_{15} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2-\delta}{A-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A-v_{\varepsilon,t})^3} \\ (6.2) \quad & \leq C_{16} + \left(C_{15} + \frac{1}{\delta} \right) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2-\delta}{A-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla (\Psi_{\varepsilon,t} + u_{\varepsilon,t}), \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A-v_{\varepsilon,t})^3}. \end{aligned}$$

Here, we used the following inequality:

$$\frac{2-\delta}{A-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla u_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \leq 2 |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}} |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}} \leq \frac{1}{\delta} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2.$$

The uniform boundedness of $\Psi_{\varepsilon,0} + u_{\varepsilon,0}$ follows from [CGP13, Section 4], Proposition 4.1 and Proposition 5.1. If $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$, by (6.2), we have the following:

$$\begin{aligned} 0 & \leq C_{16} + \left(C_{15} + \frac{1}{\delta} \right) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{\delta}{2} \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4}{(A-v_{\varepsilon,t})^3} \\ & \leq C_{16} + \left(C_{15} + \frac{1}{\delta} \right) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{\delta}{2} \frac{1}{(A+C_5)^3} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^4 \quad \text{at } (x_0, t_0). \end{aligned}$$

Therefore there exists a constant $C_{17} > 0$ satisfying

$$|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \leq C_{17} \quad \text{at } (x_0, t_0),$$

which does not depend on ε and t . By using the definition of $\Psi_{\varepsilon,t}$, $A - v_{\varepsilon,t} > 1$, and Proposition 5.1, we have the uniform upper bound of $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$ on $X \times [0, T)$, and therefore we obtain the uniform upper bound of $|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2$. \square

7. THE LAPLACIAN ESTIMATE FOR $v_{\varepsilon,t}$

In this section, we estimate $\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}$. In order to prove the uniform upper boundedness of $\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}$, we need the lower boundedness of the scalar curvature. It is obtained by [Edw15, Corollary 4.3], which is proved by the maximum principle argument as in the case of normalized Kähler-Ricci flow.

Proposition 7.1 ([Edw15, Corollary 4.3]). *The scalar curvature $R(\omega_{\varepsilon,t})$ evolves as*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) (R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon})) = |\text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}|_{\omega_{\varepsilon,t}}^2 + (R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon})),$$

and is uniformly bounded from below by

$$R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon}) \geq -C_{18},$$

where $C_{18} > 0$ is a constant independent of ε and t .

Using this estimate, we can easily obtain the following upper bound.

Proposition 7.2. *There exists a uniform constant $C_{19} > 0$ which is independent of ε and t such that*

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \leq C_{19}.$$

Proof. By Proposition 3.1, 7.1, and $u_{\varepsilon,t} \geq 0$, we have

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} = ne^{t-T} - u_{\varepsilon,t} - (1 - e^{t-T})(R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon})) \leq n + C_{18} =: C_{19},$$

which proves the assertion. \square

Proposition 7.3. *There exists a constant $C_{20} > 0$ independent of ε and t such that*

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \geq -\frac{C_{20}}{T-t}.$$

Proof. As in [Zha10, Section 3.3], we set

$$\Phi_{\varepsilon,t} := \frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}},$$

where $B > 0$ is a sufficiently large uniform constant satisfying $B - C_{19} > 0$, and $B - C_5 > 1$ so that the numerator and the denominator of $\Phi_{\varepsilon,t}$ are positive. Straightforward calculations show that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\ (7.1) \quad &= \frac{-1}{B - v_{\varepsilon,t}} \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \end{aligned}$$

$$(7.2) \quad - \frac{1}{B - v_{\varepsilon,t}} \left(\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} \right)$$

$$\begin{aligned} (7.3) \quad &+ \frac{2}{(B - v_{\varepsilon,t})^2} \text{Re} \langle \nabla v_{\varepsilon,t}, \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\ &- \frac{2}{(B - v_{\varepsilon,t})^3} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2. \end{aligned}$$

By using $B - v_{\varepsilon,t} > 1$, $B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} > 0$, and Proposition 5.1, (7.1) is estimated as follows:

$$\begin{aligned}
& \frac{-1}{B - v_{\varepsilon,t}} \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&= \left(\frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}} + \frac{-B}{B - v_{\varepsilon,t}} \right) + \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&\leq \frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}} + \frac{C_{10}}{(B - v_{\varepsilon,t})^2} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&\leq C_{21} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}),
\end{aligned}$$

where $C_{21} := 1 + C_{10} > 0$.

We next estimate (7.2). By using Lemma 5.3 (a) and Proposition 3.1 (a), we obtain

$$\begin{aligned}
& -\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} \\
&\leq -\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + (C_{22} - \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}}) \\
&= -\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} + C_{22} \\
&= \frac{1}{1 - e^{t-T}} |\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 - \frac{e^{t-T}}{1 - e^{t-T}} (\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + u_{\varepsilon,t}) + C_{22},
\end{aligned}$$

where $C_{22} := C_Z C_{10}^2 + (1 - \beta) C_1 C_{10}^2$. The first term is estimated as follows:

$$\begin{aligned}
|\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 &= |\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 + 2 \text{Re} \langle \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} \\
&\leq |\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + |\widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 + \delta |\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \frac{1}{\delta} |\widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 \\
&= (1 + \delta) |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + (1 + 1/\delta) |\widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 \\
&\leq (1 + \delta) |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + (1 + 1/\delta) C_{10}^2,
\end{aligned}$$

where $\delta > 0$ is a uniform constant determined later. Here, we used $|\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 = |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2$, $|\widehat{\omega}_T|_{\omega_{\varepsilon,t}}^2 \leq \text{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)^2 = u_{\varepsilon,t}^2 \leq C_{10}^2$. For the second term, we have

$$-\frac{e^{t-T}}{1 - e^{t-T}} (\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + u_{\varepsilon,t}) = \frac{e^{t-T}}{1 - e^{t-T}} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) - \frac{B e^{t-T}}{1 - e^{t-T}} \leq \frac{1}{1 - e^{t-T}} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}).$$

Finally, we get

$$\begin{aligned}
& -\frac{1}{B - v_{\varepsilon,t}} (\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t}) \\
&\leq \frac{C_T}{T - t} \left(\frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \left(1 + \frac{1}{\delta} \right) C_{10}^2 \right) + \frac{C_T}{T - t} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_{22},
\end{aligned}$$

where $C_T > 0$ is a uniform constant satisfying

$$\frac{1}{1 - e^{t-T}} \leq \frac{C_T}{T - t}$$

for $0 \leq t < T$. Since, we have

$$\nabla \Phi_{\varepsilon,t} = \frac{(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t})}{(B - v_{\varepsilon,t})^2} \nabla v_{\varepsilon,t} - \frac{1}{B - v_{\varepsilon,t}} \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t},$$

(7.3) can be computed as follows:

$$\begin{aligned} & \frac{2}{(B - v_{\varepsilon,t})^2} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\ &= -\frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + \frac{2}{(B - v_{\varepsilon,t})^3} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2. \end{aligned}$$

Combining these estimates, we get

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\ & \leq C_{21} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\ & \quad + \frac{C_T}{T-t} \left(\frac{1+\delta}{B - v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \left(1 + \frac{1}{\delta} \right) C_{10}^2 \right) + \frac{C_T}{T-t} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_{22} \\ & \quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\ & \leq \frac{C_{23}}{T-t} + \frac{C_{23}}{T-t} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + \frac{C_T}{T-t} \frac{1+\delta}{B - v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) (T-t) \Phi_{\varepsilon,t} = -\Phi_{\varepsilon,t} + (T-t) \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \leq (T-t) \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\ & \leq C_{23} + C_{23} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_T \frac{1+\delta}{B - v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla (T-t) \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}. \end{aligned}$$

We set $\tilde{\Psi}_{\varepsilon,t} := \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{B - v_{\varepsilon,t}}$. Combining with Lemma 6.2 (c) and (6.1), we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \tilde{\Psi}_{\varepsilon,t} \\ & \leq C_{24} - \frac{|\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{B - v_{\varepsilon,t}} + \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \tilde{\Psi}_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}, \end{aligned}$$

where $C_{24} := C_{14} + (1-\beta)C_1C_{14} + C_{14}C_{10} > 0$. On the other hand, we have

$$\frac{4}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \leq \delta |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 + \frac{4}{\delta} |\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2.$$

By using Lemma 5.3 (b), Proposition 5.1, and Proposition 6.1, we get

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) u_{\varepsilon,t} \leq u_{\varepsilon,t} + C_{11} u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}} \\ & \leq C_{10} + C_{11} C_{10}^2 - \frac{1}{C_{10}} |\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \\ & \leq -\frac{4}{B - v_{\varepsilon,t}} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + C_{25}, \end{aligned}$$

where we take $0 < \delta < 1/C_{10}$, and $C_{25} := C_{10} + C_{11}C_{10}^2 + 4C_{14}/\delta > 0$.

Combining these inequalities, we have

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \left((T-t)\Phi_{\varepsilon,t} + 2C_T\tilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t} \right) \\
& \leq C_{23} + C_{23}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) + C_T \frac{1+\delta}{B-v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 - \frac{2}{B-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla (T-t)\Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\
& \quad + 2C_T \left(C_{24} - \frac{|\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{B-v_{\varepsilon,t}} + \frac{2}{B-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{2}{B-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla \tilde{\Psi}_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \right) \\
& \quad + 2C_T \left(-\frac{4}{B-v_{\varepsilon,t}} \operatorname{Re} \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + C_{25} \right) \\
& \leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T \frac{1-\delta}{B-v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \\
& \quad - \frac{2}{B-v_{\varepsilon,t}} \operatorname{Re} \left\langle \nabla \left((T-t)\Phi_{\varepsilon,t} + 2C_T\tilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t} \right), \nabla v_{\varepsilon,t} \right\rangle_{\omega_{\varepsilon,t}}.
\end{aligned}$$

The uniform boundedness of $(T-t)\Phi_{\varepsilon,t} + 2C_T\tilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}$ at $t=0$ follows from [CGP13, Section 4], Proposition 4.1, Proposition 5.1 and Proposition 6.1. If $(T-t)\Phi_{\varepsilon,t} + 2C_T\tilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$, we have the following at this point:

$$\begin{aligned}
0 & \leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T \frac{1-\delta}{B-v_{\varepsilon,t}} |\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 \\
& \leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T \frac{1-\delta}{B+C_5} \left(\frac{1}{n} (B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t})^2 - \frac{B^2}{n} \right).
\end{aligned}$$

Here, we used Proposition 4.1, and

$$|\nabla \bar{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2 = g^{i\bar{j}} g^{k\bar{l}} (\partial_i \partial_{\bar{j}} v_{\varepsilon,t}) (\partial_{\bar{k}} \partial_l v_{\varepsilon,t}) \geq \frac{1}{n} (\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t})^2 \geq \frac{1}{n} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t})^2 - \frac{B^2}{n}.$$

Therefore, at this point, there exists a constant C_{27} satisfying

$$-\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \leq C_{27} \quad \text{at } (x_0, t_0)$$

which is independent of ε , t , and (x_0, t_0) . Combining Proposition 4.1, Proposition 6.1, and Proposition 5.1, we obtain the uniform upper boundedness of $(T-t)\Phi_{\varepsilon,t} + 2C_T\tilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}$ on $X \times [0, T]$. Finally, we conclude that

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \geq -\frac{C_{20}}{T-t}$$

□

Proof of Theorem A By Proposition 3.1, and Proposition 7.3, we have

$$\begin{aligned}
R(\omega_{\varepsilon,t}) - \operatorname{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon}) &= \frac{1}{1-e^{t-T}} (-\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + ne^{t-T} - u_{\varepsilon,t}) \\
&\leq \frac{C_T}{T-t} \left(\frac{C_{20}}{T-t} + n \right) \leq \frac{C}{(T-t)^2},
\end{aligned}$$

where $C > 0$ does not depend on ε and t . Therefore, by taking $\varepsilon_i \rightarrow 0$, we get the assertion. □

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO
 3-8-1 KOMABA, MEGURO-KU, TOKYO, 153-8914, JAPAN.
E-mail address: nomu@ms.u-tokyo.ac.jp